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The spatial model with non-policy factors: a theory of policy-motivated candidates

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Abstract The theoretical literature on two candidate elections is dominated by symmetric contests and vote-maximizing candidates. These models fail to capture two important features of real elections. First, most elections pit a stronger candidate against a weaker one. Second, candidates care not only about holding office, but also about policy outcomes. Ignoring any one of these features means we will fail to capture an important dynamic—strong candidates must balance their desire to change policy with their need to win the election. We provide conditions for the existence of an equilibrium in the spatial model with non-policy factors, when candidates are policy motivated. We provide a characterization of 'regular' equilibria and show that there exists at most one regular equilibrium. We provide conditions that guarantee that all equilibria are regular. We derive comparative statics for the model and show that increasing a candidate's non-policy advantage causes that candidate to move towards his ideal point.

1 Introduction

The theoretical literature on two candidate elections is dominated by symmetric contests and vote-maximizing candidates. These models fail to capture two important features of real elections. First, most elections are not symmetric contests. Instead, they pit a strong candidate against a weaker one. This is particularly likely to be the case in legislative elections, which feature a strong incumbent and a weak challenger. Second, candidates care not only about holding office, but also about policy outcomes. Ignoring any one of these two features means we will fail to capture an

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important dynamic—strong candidates must balance their desire to change policy with their need to win the election.

Two candidate elections are a topic of central importance in political economy. The literature has realized the importance of incorporating asymmetry in electoral contests. But the shortcomings of the vote-maximizing theory limit our ability to properly theorize about two candidate elections and interpret empirical results in this area.[1] We will consider a two-candidate multi-dimensional probabilistic voting model that includes non-policy factors, and allows candidates to care about both winning office and policy outcomes. Non-policy factors affect the voters' preferences for the candidates, but the candidates cannot compete over them in the same way as they compete over policy. Non-policy factors are an extension of 'valence' (Stokes 1963).[2] Unlike valence, non-policy factors are not necessarily perceived uniformly across the population.[3] Because of this, they are general enough to include identification with a political party, incumbency (which may be perceived positively by some voters and negatively by others), retrospective evaluations of the candidates' performance, and multiple types of valence (charisma, height, good looks, etc.). Non-policy factors introduce an asymmetry into the spatial model.

When asymmetries are absent, the equilibrium in the one-dimensional spatial model is robust to the motivations of the candidates (Wittman 1983; Calvert 1985; Roemer 2001).[4] Hence, much of the literature has viewed vote-maximizing candidates as a shortcut to achieving theoretical results without unnecessary complications (Downs 1957). Asymmetries (and valence in particular) are, however, increasingly viewed as important components of spatial competition (Schofield et al. 1998; Adams et al. 2005).

1.1 Vote-maximizing candidates

The two-candidate probabilistic voting model, introduced by Hinich (1977), was originally developed as an extension to the Downsian model (Downs 1957). Hinich showed that when voters have quadratic utility functions and candidates maximize votes, both candidates will converge to the mean voter's position. Coughlin and Nitzan (1981) considered more general utility functions and showed that equilibria involve both candidates locating at the policy position which maximizes social welfare. Similar results exist for the multi-candidate case (Lin et al. 1999; McKelvey and Patty 2006).

Common to all this work is the assumption that elections are symmetric contests. Non-policy factors (or valence, or additive bias) introduce asymmetry into these

[1] Empirical models of two-candidate competition based on the spatial model include, Erikson and Romero (1990); Alvarez and Nagler (1995); Schofield et al. (1998); Martin et al. (1999); Lacy and Burden (1999); Adams and Merrill (2003); Callander and Wilson (2006), and Peress (2006).

[2] Models of spatial competition with valence include Enelow and Hinich (1982); Ansolabehere and Snyder (2000); Groseclose (2001); Aragones and Palfrey (2002); Schofield (2003, 2004) and Wittman (2005).

[3] Our spatial model with non-policy factors shares much in common with Banks and Duggan's (2005) additive bias model.

[4] Wittman, Calvert, and Roemer show that policy-motivated candidates converge in equilibrium when aggregate uncertainty is not present, but diverge when aggregate uncertainty is present.

elections. Banks and Duggan (2005) considered a framework that allowed for non-policy factors (additive bias in their terminology).[5] As Banks and Duggan demonstrated, convergent equilibria are a very general feature of two-candidate spatial competition with vote-maximizing candidates. Banks and Duggan also provided sufficient conditions for the existence and uniqueness of vote-maximizing equilibria.

Although studying policy-motivated candidates is the main objective of this paper, characterizing the vote-maximizing point is an important step in characterizing the policy-motivated equilibrium because vote-maximizing equilibria and policy-motivated equilibria correspond in symmetric races (a fact we elaborate on in this paper). We therefore provide a characterization of the unique vote-maximizing point in the spatial model with non-policy factors.

Non-policy factors introduce new problems of equilibrium existence to the vote-maximizing theory. The least restrictive conditions which are known to guarantee the existence of a vote-maximizing equilibrium require quasi-concavity of both candidates vote-share functions (Banks and Duggan 2005). We will demonstrate that vote-maximizing points will most often fail to exist when one candidate has a sufficiently strong non-policy advantage. In contrast, we will demonstrate that policy-motivated equilibria generally exist, provided that the 'advantaged' candidate's vote share function is quasi-concave. This condition is likely to be met even when one of the candidates has a strong non-policy advantage.

1.2 Policy-motivated candidates

Calvert (1985) considered policy-motivated candidates in the multidimensional Downsian model. He found that if the voting game admits a core, then an equilibrium exists, and involves both candidates locating at the core.[6] Duggan and Fey (2005) demonstrated that policy-motivated equilibria often fail to exist when the core is empty.

Groseclose (2001) examined policy-motivated candidates in the Downsian model with valence. The framework allowed for aggregate uncertainty, but the main theoretical results were restricted to the case where a candidate has a small valence advantage. As a candidate's valence advantage increased from zero to a small number, that candidate approached the political center.

Our theory expands on the existing literature in the following ways. Duggan and Fey (2005) demonstrate that many of the limitations of the Downsian model extend to the case of policy-motivated candidates. Specifically, pure strategy equilibria will often fail to exist when the core is empty, which is likely when the policy-space is multidimensional. The probabilistic-voting model has been viewed as a 'solution' to the problem of an empty core, in the case of vote-maximizing candidates. Our main result shows that this extends to the case of policy-motivated candidates. We provide conditions for the existence of an equilibrium in the spatial model with non-policy factors,

[5] See Schofield (2003, 2004) for a multi-candidate probabilistic voting model with valence.

[6] A core point is a position in the policy space that (weakly) defeats any other position in the policy space. A well known result states that a core generally does not exist when the policy dimension is greater than one (Plott 1967; McKelvey 1976; Schofield 1978).

when candidates are policy-motivated. Consistent with Duggan and Fey's result, our existence result requires that the pseudo-core is non-empty. The pseudo-core is an extension of the core to the spatial model with non-policy factors.

We provide a characterization of 'regular' equilibria, which are equilibria where the candidate that has a non-policy advantage locates as close as possible to his ideal point, while still winning the election. We show that there exists at most one regular equilibrium. We provide conditions that guarantee that all equilibria are regular. In addition, we provide comparative static results. The most interesting comparative static result concerns the effect of the size of the non-policy advantage on the position of the advantaged candidate. We find that while a strong candidate must temper his policy proposals in order to satisfy his constituents, his advantage on the non-policy dimension allows him to propose policies that deviate from the political center. He is able to move the policy outcome away from the vote-maximizing point and towards his own ideal point. The advantaged candidate will position as close to his ideal point as possible while still winning the election. This balancing of personal policy preferences and the need to secure reelection is a critical feature of elections (particularly legislative elections) not characterized by existing spatial models of elections.

2 The model

We assume there are two candidates competing for office—candidate L and candidate R.[7] Policy is characterized using the J-dimensional spatial model. We denote the positions of candidate L and candidate R by $y_L \in Y$ and $y_R \in Y$ where $Y \subseteq \mathbb{R}^J$ is the set of positions that the candidates are allowed to take. Voters are completely characterized by their ideal points $v \in \mathbb{R}^J$ and by their non-policy factors $(z_L, z_R) \in \mathbb{R}^2$. There is a continuum of voters represented by the density function $f(v, z_L, z_R)$.[8]

2.1 Voter objectives

The utility that a voter characterized by (v, z_L, z_R) receives from voting for candidate k is represented by $u(y_k - v, z_k) = z_k + h(y_k - v)$ where $h : \mathbb{R}^J \to \mathbb{R}$. For expositional convenience, we will write $z = z_L - z_R$ and replace the distribution $f(v, z_L, z_R)$ with the distribution $f(v, z)$ since only the difference $z_L - z_R$ matters in terms of partitioning the set of voters among the candidates. The set of voters that strictly prefer the left-wing candidate can be characterized by,

$$\{(v, z) : u(y_L - v, z_L) > u(y_R - v, z_R)\} = \{(v, z) : z > h(y_R - v) - h(y_L - v)\}$$

[7] For expositional purposes, we will think of L and R as the left-wing and right-wing candidates, even though the model is potentially multidimensional.

[8] We assume that $f(v, z_L, z_R)$ represents the subpopulation of voters rather than the entire population. The restriction of full turnout is without loss of generality so long as the voters' turnout decisions do not depend on the positions that the candidates take.

We can write the vote shares of the candidates at positions y_L and y_R as,

$$s_L(y_L, y_R) = \int_{v\in\mathbb{R}^J} \int_{z=h(y_R-v)-h(y_L-v)}^{\infty} f(v,z)\mathrm{d}z\mathrm{d}v$$

$$s_R(y_L, y_R) = \int_{v\in\mathbb{R}^J} \int_{z=-\infty}^{h(y_R-v)-h(y_L-v)} f(v,z)\mathrm{d}z\mathrm{d}v$$

We assume that the following properties hold:

(A1) Y is convex and compact.
(A2) $h(x)$ is second-order continuously differentiable for $x \in \mathbb{R}^J$.
(A3) $\frac{\partial^2}{\partial x \partial x^T} h(x)$ is negative definite for all $x \in \mathbb{R}^J$.
(A4) $f(v,z) > 0$ and $f(v,z)$ is continuously differentiable for all $(v,z) \in \mathbb{R}^{J+1}$.
(A5) There exists a $y^* \in \mathrm{int}(Y)$ such that $\int_{v\in\mathbb{R}^J} \frac{\partial h}{\partial x}(y^* - v) f_{v|z}(v|0)\mathrm{d}v = 0$.[9]

A particularly interesting choice for the utility function is the quadratic specification,

$$u(y-v,z) = z - \frac{1}{2}(y-v)^T A(y-v)$$

where A is positive semi-definite. Assumptions (A2) and (A3) are clearly satisfied by the quadratic utility function. It is standard in the probabilistic voting literature to assume (A5) as a primitive condition.[10] For the quadratic utility function, (A5) is equivalent to $y^* = \int_{v\in\mathbb{R}^J} v f_{v|z}(v|0)\mathrm{d}v = E[v|z=0] \in Y$. Hence, Assumption (A5) will be satisfied if the mean ideal point for voters with a non-policy factor of zero exists and is located in Y. The assumption that the mean of the distribution exists is clearly weak. The requirement that the mean is located in Y is purely technical because the set Y can be expanded to accommodate this requirement. Assumption (A5) will also be satisfied for the one-dimensional power utility function, $h(x) = |x|^\delta$ when $\delta > 1$, under similar conditions.[11]

We refer to the special case where v and z are independent and z is mean zero as the *standard probabilistic voting model* (Coughlin and Nitzan 1981). The situation where v and z are independent, but z is not necessarily mean zero will be referred to as the *probabilistic voting with valence model* (Schofield 2003, 2004). We will refer to the general case as the *spatial model with non-policy factors*.

9 In Proposition 2, we will show that y^* is unique.

10 See McKelvey and Patty (2006) and Banks and Duggan (2005).

11 The case where $\delta \leq 1$ will violate assumption (A2).

2.2 Candidate objectives

We assume that candidate $k \in \{L, R\}$ has a utility function of the form, $U_k(x, w_k)$, where x denotes the policy outcome (determined by the candidate that is elected to office) and $w_k \in \{0, 1\}$ indicates whether candidate k wins the election. We further assume that,

(B1) $U_k(x, w_k)$ is continuously differentiable and strictly concave in x for $k \in \{L, R\}$.
(B2) $U_k(x, 1) > U_k(x, 0)$ for all $x \in Y$ and $k \in \{L, R\}$.
(B3) There exists a unique $q_k \in \text{int}(Y)$ such that $q_k = \arg\max_{x \in Y} U_k(x; 1)$ for $k \in \{L, R\}$.
(B4) $q_k \neq y^*$ for $k \in \{L, R\}$.

Here, assumption (B1) will be used for existence and uniqueness results. Assumption (B2) indicates that holding office is in it of itself valuable to the candidates. Assumption (B3) requires that the candidates have unique ideal points and that the set of admissible policies is large enough to include the ideal points of the candidates. Assumption (B4) requires that the election be 'competitive' in the sense that neither candidate is fully satisfied with the vote-maximizing outcome.

2.3 Defining equilibrium

Before we can proceed, we must deal with a technical problem. Suppose that we were to assume that in case of a tie, each candidate is chosen as the winner with equal probability. This would lead to the 'open-set problem' described in Groseclose (2001).[12] Essentially, since the candidates' utility functions depend on the identity of the winner and the identity of the winner is not a continuous function of the positions of the candidates, this can lead to a situation where an equilibrium fails to exist. Groseclose (2001) proposed to solve the 'open-set' problem by introducing aggregate uncertainty. We will take a different approach. Following Simon and Zame (1990) and Duggan (2006), we will consider 'endogenous sharing rules'. Specifically, we incorporate the tie-breaking rule into the definition of the equilibrium. This will allow us to replicate the limiting equilibria that result from considering small amounts of aggregate uncertainty, without introducing uncertainty explicitly. As Simon and Zame argue, this also allows us to replicate the limiting equilibria that would occur in comparable finite action games.[13]

Let $t \in [0, 1]$ represent the probability that candidate L is selected as the winner in case $s_L(y_L, y_R) = \frac{1}{2}$ (i.e., in case of a tie). The candidates' utility functions over policy positions are given by,

[12] Duggan (2006) discusses the open-set problem in the context of legislative agendas.

[13] Aragones and Palfrey (2002) solve the open set problem by directly considering the limiting equilibria of a sequence of finite action games in the context of vote-maximizing candidates in the Downsian model with aggregate uncertainty and valence.

$$V_L(y_L, y_R, t) = \begin{cases} U_L(y_L, 1), & s_L(y_L, y_R) > \frac{1}{2} \\ tU_L(y_L, 1) + (1-t)U_L(y_R, 0), & s_L(y_L, y_R) = \frac{1}{2} \\ U_L(y_R, 0), & s_L(y_L, y_R) < \frac{1}{2} \end{cases}$$

$$V_R(y_L, y_R, t) = \begin{cases} U_R(y_L, 0), & s_L(y_L, y_R) > \frac{1}{2} \\ tU_R(y_L, 0) + (1-t)U_R(y_R, 1), & s_L(y_L, y_R) = \frac{1}{2} \\ U_R(y_R, 1), & s_L(y_L, y_R) < \frac{1}{2} \end{cases}$$

It is the discontinuity of V_k in the policy positions at points such that $s_L(y_L, y_R) = \frac{1}{2}$ that causes equilibrium existence problems (Simon and Zame 1990).[14] We define an equilibrium as pair of points in the policy space $(y_L^*, y_R^*) \in Y^2$ and a tie-breaking rule $t^* \in [0, 1]$ such that y_L^* maximizes $V_L(y_L, y_R^*, t^*)$ over $y_L \in Y$ and y_R^* maximizes $V_R(y_L^*, y_R, t^*)$ over $y_R \in Y$.

We will define an equilibrium (y_L^*, y_R^*, t^*) as *non-satiated* if $y_L^* \neq q_L$ and $y_R^* \neq q_R$. Otherwise, we say the equilibrium is *satiated*. Duggan and Fey (2005) found that this property of equilibria was important in their study of policy-motivated equilibria in the Downsian model.

2.4 Advantaged candidates

When candidates are policy-motivated, the qualitative properties of the equilibrium will depend crucially on which candidate has a *non-policy advantage*, a concept we will now discuss. Let us define,

$$\tilde{z} = \int\limits_{v \in R^J, z>0} f(v, z) \mathrm{d}v \mathrm{d}z - \int\limits_{v \in R^J, z<0} f(v, z) \mathrm{d}v \mathrm{d}z$$

If $\tilde{z} > 0$, we say that candidate L has a non-policy advantage. If $\tilde{z} < 0$, we say that candidate R has a non-policy advantage. Otherwise, we say that neither candidate has a non-policy advantage. Here, $\tilde{z}$ captures the fraction of voters that would vote for the left-wing candidate minus the fraction of voters that would vote for the right wing candidate, if both candidates located at the same position.

If one candidate has a non-policy advantage, then that candidate would win if both candidates were to locate at the same position. To see that this is the case, suppose that $y_L = y_R = y$. Then,

$$s_L(y, y) - s_R(y, y) = \int\limits_{v \in R^J, z>0} f(v, z) \mathrm{d}v \mathrm{d}z - \int\limits_{v \in R^J, z<0} f(v, z) \mathrm{d}v \mathrm{d}z = \tilde{z}$$

Thus, $s_L(y, y) > s_R(y, y)$ if $\tilde{z} > 0$ and $s_L(y, y) < s_R(y, y)$ if $\tilde{z} < 0$.

[14] Recall that continuity was a requirement of the Debreu–Fan–Glicksberg Theorem.

3 Vote-maximizing points

We define a *vote-maximizing point* as a pair of positions $(y_L^*, y_R^*) \in Y^2$ such that $y_L^* \in \arg\max_{y_L \in Y} s_L(y_L, y_R^*)$ and $y_R^* \in \arg\max_{y_R \in Y} s_R(y_L^*, y_R)$. We say that a vote-maximizing point is *interior* if $y_L^* \in \text{int}(Y)$ and $y_R^* \in \text{int}(Y)$. This section focuses on characterizing these points. Characterizing such points is an important step in characterizing policy-motivated equilibrium.

We will show that an interior vote-maximizing point must involve both candidates converging to the same point y^*, which is characterized by the proposition below.

Proposition 1 *Suppose that assumptions* (A1–A5) *hold. Then any interior vote-maximizing point must satisfy,*

(i) $y_L^* = y_R^* = y^*$.
(ii) $\int_{v \in \mathbb{R}^J} \frac{\partial h}{\partial x}(y^* - v) f_{v|z}(v|0)\mathrm{d}v = 0$.

While Proposition 1 characterizes the vote-maximizing point when it does exist, we would like to know if a unique vote-maximizing point exists. The next proposition provides conditions under which this is the case.

Proposition 2 *Suppose that assumptions* (A1–A5) *hold.*

(i) *There exists at most one interior vote-maximizing point.*
(ii) *If $s_L(y_L, y_R)$ is quasi-concave in y_L and $s_R(y_L, y_R)$ is quasi-concave in y_R, then a vote-maximizing point exists.*

A unique vote-maximizing point is guaranteed in our framework. Verifying the existence of a vote-maximizing point is more difficult. Quasi-concavity is a difficult condition to check, even for relatively simple choices of f and h. In Sect. 7, we provide conditions under which the vote share functions are concave (and consequently, quasi-concave). Below, we provide an example that suggests that a vote-maximizing point is unlikely to exist when one of the candidates has a strong non-policy advantage.

We can examine whether condition (ii) of Proposition 2 is likely to be satisfied by considering the quadratic utility function, which has $h(x) = -\frac{1}{2}x^T Ax$ where A is symmetric and positive definite. If an equilibrium exists, it has the form $y_L = y_R = y^*$ where,

$$y^* = \int\limits_{v \in \mathbb{R}^J} v f_{v|z}(v|0)\mathrm{d}v$$

The second-order conditions from this problem require that the following matrices are negative definite,

$$M_L = -f_z(0)A - A\left[\int\limits_{v \in \mathbb{R}^J} (y^* - v)(y^* - v)^T f_z'(0|v) f_v(v)\mathrm{d}v\right] A^T$$

$$M_R = -f_z(0)A + A\left[\int\limits_{v \in \mathbb{R}^J} (y^* - v)(y^* - v)^T f_z'(0|v) f_v(v)\mathrm{d}v\right] A^T$$

If we assume that $J = 1, A = [a]$, and $f(v, z) = f_v(v) f_z(z)$, we find that the following condition is required:

$$|f_z'(0)| < \frac{1}{a\sigma^2} f_z(0)$$

where $\sigma^2 = \int_{v\in\mathbb{R}} (y^* - v)^2 f_v(v)\mathrm{d}v$. If z is normally distributed with mean λ and variance 1, we get $|\lambda| < \frac{1}{a\sigma^2}$. If one candidate has a strong non-policy advantage (e.g. $|\lambda| > \frac{1}{a\sigma^2}$), the second-order condition will fail to hold for at least one candidate. Hence, y^* will not be a local vote-maximizing point. A local vote-maximizing point will fail to exist if one candidate's non-policy advantage ($|\lambda|$) is sufficiently large relative to the importance of policy voting ($\frac{1}{a\sigma^2}$) because while the disadvantaged candidate maximizes votes by moving away from the advantaged candidate, the advantaged candidate maximizes votes by moving towards the disadvantaged candidate. Since (y^*, y^*) is the only possible vote-maximizing point (see Proposition 1), a vote-maximizing point will fail to exist if one candidate has a sufficiently strong non-policy advantage.[15]

4 The left-wing candidate has a non-policy advantage

We begin by characterizing the equilibrium where the left-wing candidate has a non-policy advantage, i.e., $\tilde{z} > 0$. Results for the case where $\tilde{z} < 0$ can be derived symmetrically. The case where neither candidate has a non-policy advantage (i.e. $\tilde{z} = 0$) will be considered in the next section.

The candidate with a non-policy advantage must win in equilibrium with probability one in an interior non-satiated equilibrium. If the advantaged candidate does not locate at his ideal point, then the tie-breaking rule must select the advantaged candidate with probability one.

Proposition 3 *Suppose that* $\tilde{z} > 0$ *and that assumptions* (A1–A5) *and* (B1–B4) *hold. Any interior policy-motivated equilibrium* (y_L^*, y_R^*, t^*) *must satisfy either* (*a*) $s_L(y_L^*, y_R^*) = \frac{1}{2}$, *or* (*b*) $s_L(y_L^*, y_R^*) \geq \frac{1}{2}$ *and* $y_L^* = q_L$. *If the equilibrium is non-satiated, then* $t^* = 1$.

Proposition 3 demonstrates why we need to consider deterministic tie-breaking rules. If deterministic tie-breaking rules are disallowed, then an interior non-satiated equilibrium will fail to exist. As we argue later, satiated equilibria are unlikely to exist under realistic conditions. Hence, without deterministic tie-breaking rules, the

[15] We note that this equilibrium existence failure is different in nature from the one characterized in Aragones and Palfrey (2002). The failure of a pure strategy equilibrium to exist in Aragones and Palfrey is due to the coarseness of the policy space. They consider a coarse policy space to deal with the open set problem which results when once candidate has a valence advantage. The open set problem does not occur in the probabilistic voting model with vote-maximizing candidates, and hence does not apply to our framework.

theory of policy-motivated candidates would be of little use because equilibrium non-existence would be prevalent.[16]

Define $W_L = \{y_L \in Y : \min_{y_R \in Y} s_L(y_L, y_R) \geq \frac{1}{2}\}$. We will refer to this set as the *pseudo-core*. This set will reduce to the core when non-policy factors are absent from the model (i.e. z contains a point mass at zero). We will focus on a particular type of policy-motivated equilibrium. We say a policy-motivated equilibrium is *regular* if $y_L^* \in W_L$. All other equilibria will be called *irregular*. In a regular equilibrium, the advantaged candidate will position himself as close to his ideal point as he can while still winning the election and the disadvantaged candidate will respond by maximizing his vote share.

Proposition 4 *Suppose that* $\tilde{z} > 0$ *and that assumptions* (A1–A5) *and* (B1–B4) *hold. If* (y_L^*, y_R^*, t^*) *is a regular policy-motivated equilibrium, then* $y_L^* \in \arg\max_{y_L \in W_L} U_L(y_L, 1)$. *If, in addition,* $y_L^* \neq q_L$, *then* $y_R^* \in \arg\max_{y_R \in Y} s_R(y_L^*, y_R)$.

Proposition 4 shows that regular equilibria are equilibria where the loosing candidate cannot win by repositioning.

An alternative type of equilibrium is possible. Under an irregular equilibrium (with $y_L^* \notin W_L$), the losing candidate could win the election by moving, but does not, because he prefers the current policy outcome. While these equilibria are possible, we argue that they are unlikely to exist under realistic conditions. Proposition 5 shows that an irregular non-satiated equilibrium will not exist when $s_L(y_L, y_R)$ is quasi-concave in the first argument and quasi-convex in the second. If $q_L \in W_L$, then the equilibrium must satisfy $y_L^* = q_L$. Hence, an irregular equilibrium cannot exist if the advantaged candidate has either a very small or a very large non-policy advantage. If candidates are sufficiently office-motivated, then we cannot have an equilibrium with $y_L^* \notin W_L$.[17] Finally, we extensively checked various examples in the one-dimensional model by computing equilibria numerically. While we found that irregular equilibria did occasionally exist, they appeared under unrealistic conditions. We found that irregular equilibria existed only when both candidates' ideal points were located on the same side of most of the voter ideal points (in addition to the conditions already mentioned).

The result below provides conditions for the existence and uniqueness of regular equilibria, and provides conditions under which irregular non-satiated equilibria cannot exist.

Proposition 5 *Suppose that* $\tilde{z} > 0$ *and that assumptions* (A1–A5) *and* (B1–B4) *hold. Suppose that* ∂W_L *is a* C^1 *manifold in a neighborhood of* y_L^*.[18] *If* $\text{int}(W_L) \neq \emptyset$ *and* $s_L(y_L, y_R)$ *is quasi-concave in its first argument, then there exists a regular*

[16] Of course, we are not saying that deterministic tie-breaking rules should be allowed simply because an equilibrium would otherwise fail to exist. Our point is that the failure of equilibrium to exist is purely technical is this case, and thus a purely technical correction is appropriate.

[17] The argument here is straightforward. If $y_L^* \notin W_L$, then there exists a $y_R \in Y$ such that $s_L(y_L^*, y_R) < \frac{1}{2}$. Moving to y_R will increase candidates R's utility if he is sufficiently office-motivated.

[18] We use ∂A to denote the boundary of a set A. A C^1-manifold is a manifold that is continuously first-order differentiable (Mas-Colell 1985).

policy-motivated equilibrium. Candidate L's position is uniquely determined in a regular policy-motivated equilibrium. The equilibrium (y_L^*, y_R^*, t^*) *exhibits divergence and satisfies* $U_L(y_R^*, 1) < U_L(y_L^*, 1)$. *If* $s_L(y_L, y_R)$ *is quasi-convex in its second argument, then* $\text{int}(W_L) \neq \emptyset$ *and no interior irregular non-satiated policy-motivated equilibrium exists.*

Proposition 5 requires three conditions in addition to those assumed throughout the paper. First, ∂W_L must be smooth. We cannot guarantee that this will be the case simply by placing restrictions on the vote share functions.[19] Though hard to check, we believe the condition to be a weak one as the Envelope Theorem implies that the set of discontinuities of the extreme value function will have measure zero.[20]

The second condition is that the advantaged candidate's vote share function is quasi-concave in his own position. We argued in Sect. 3 that vote-maximizing equilibria will typically fail to exist when one candidate has a large non-policy advantage. This occurs because at the unique point which satisfies both candidates' first-order conditions, the disadvantaged candidate's second-order condition is not satisfied. Our existence result for policy-motivated candidates requires quasi-concavity only for the advantaged candidate. This assumption is much more likely to hold, and hence a policy-motivated equilibrium is much more likely to exist when one of the candidates is strong.

The third condition requires that the set $\text{int}(W_L)$ be non-empty. In comparison to our result, Duggan and Fey (2005) argue that policy-motivated equilibria often fail to exist in the deterministic voting model when the core is empty. Their result is consistent with our result because in the deterministic voting model, W_L will be equal to the core. Hence, if the core is empty, our result would not apply to deterministic voting models because W_L would be empty. The existence of vote-maximizing equilibria is known to be less of a problem in the multi-dimensional probabilistic voting model. Our results show that this extends to policy-motivated equilibria. Moreover, these equilibria are robust to shifting the distribution of z to allow one candidate to have a non-policy advantage.

Although the equilibrium outcome and the position of the winning candidate are generally pinned down by the equilibrium, the position of the losing candidate is not necessarily pinned down. For example, consider the case where the winning candidate has a non-policy advantage that is so strong that he is able to locate at his ideal point. In this case, any position of the losing candidate will result in equilibrium. Even when $y_L^* \neq q_L$, the set $\left\{y_R \in Y : s_L(y_L^*, y_R) = \frac{1}{2}\right\}$ is unlikely to contain a unique element when $J > 1$. We do not view this as a serious problem for the theory because the position of the losing candidate can be pinned down in other ways. For example, we can assume that if his current motivations do not fully constrain him, the losing candidate will locate as close as possible to his policy ideal point.

[19] Define $\phi(y_L) = \min_{y_R \in Y} s_L(y_L, y_R)$. Danskin (1966) shows that $\phi(y_L)$ need not be differentiable even if $s_L(y_L, y_R)$ is infinitely differentiable in both arguments. The counter example he gives is quasi-convex in its first argument. Concavity may be enough to ensure differentiability (Milgrom and Segal 2002), but this would rule out many cases on interest.

[20] Contrarily, differentiability of $U_L(y_L, 1)$ is sufficient to ensure that $\partial\{x : U_L(x, 1) \geq U_L(y_L^*, 1)\}$ will be smooth, which is necessary for the second application of Lemma 2.

5 Neither candidate has a non-policy advantage

In this section, we consider the case where neither candidate has a non-policy advantage (e.g. $\tilde{z} = 0$). All interior non-satiated equilibria will have both candidates locating at the vote-maximizing point y^*.

Proposition 6 *Suppose that* $\tilde{z} = 0$ *and that assumptions* (A1–A5) *and* (B1–B4) *hold. If* (y_L^*, y_R^*, t^*) *is an interior non-satiated policy-motivated equilibrium, then* $s(y_L^*, y_R^*) = \frac{1}{2}$. *If in addition* $t^* \in (0, 1)$, *then* $y_L^* = y_R^* = y^*$ *must hold in equilibrium.*

Notice that this result is consistent with Theorem 1 of Duggan and Fey (2005), which concerns policy-motivated equilibria in the multi-dimensional Downsian model. We find, as they do, that all interior non-satiated equilibria exhibit policy convergence.[21] Satiated equilibria are possible as well, but we will argue that they are unlikely to occur for reasons we explain below.

By imposing quasi-concavity on both candidates' vote share functions, we can obtain a stronger result, that will guarantee the existence and uniqueness of policy-motivated equilibrium when $\tilde{z} = 0$. Although we argued that this condition was unlikely to be met when one candidate had a large non-policy advantage, it is reasonable to apply it here.

Proposition 7 *Suppose that* $\tilde{z} = 0$ *and that assumptions* (A1–A5) *and* (*B1–B4*) *hold. Suppose that* $s_L(y_L, y_R)$ *is quasi-concave* y_L *and quasi-convex in* y_R. *There exists a unique interior policy-motivated equilibrium with* $y_L = y_R = y^*$.

It is worth pointing out some differences between our results and Duggan and Fey (2005). First, while Duggan and Fey (2006) show that under certain conditions, an equilibrium does not exist, we provide conditions under which an equilibrium exists. The added condition we impose is that the vote-share function is appropriately quasi-concave. Duggan and Fey consider the situation where a core does not exist. By the Debreu–Fan–Glicksberg theorem, the failure of a core to exist implies that the vote share functions are not both quasi-concave if vote share functions are continuous. Duggan and Fey are chiefly concerned with policy-motivated equilibria in the Downsian model while we are chiefly concerned with policy-motivated equilibria in the probabilistic voting model. Paralleling findings from the vote-maximizing case, Duggan and Fey find that equilibria will often fail to exist in the Downsian model when a core does not exist, while we find that equilibria typically exist in the probabilistic voting model.

6 Comparative statics

Now that we have characterized the possible policy-motivated equilibria, we can begin to derive comparative statics. Since we argued that irregular equilibria were unlikely

[21] Duggan and Fey only consider non-deterministic tie-breaking rules.

to occur, we will focus on regular equilibria. There are three important parameters that we can vary. We can vary the non-policy advantage of the candidates by shifting around the distribution of z. We can vary the mean voter by shifting around the distribution of v. We can vary the ideal points of the candidates as well. We will consider the effect of each of these in this subsection.

Having a non-policy advantage is beneficial to a candidate, in the sense that he is able to secure a more favorable policy outcome than that which occurs under the vote-maximizing equilibrium. If a candidate has a non-policy advantage, then that candidate is able to move the policy outcome away from the vote-maximizing equilibrium and towards his ideal point. Increasing a candidate's non-policy advantage in a uniform way will always allow that candidate to move closer to his ideal point. Moreover, if the candidate's non-policy advantage is strong enough, he will position at his ideal point. This logic is captured in the following result.

Proposition 8 *Consider a collection of models indexed by* λ *where* $f(v, z) = g(v, z - \lambda)$ *with* $\tilde{z}(0) = 0$*. Suppose that assumptions* (A1–A5) *and* (*B1–B4*) *hold for all* λ*. Let* $X^*(\lambda)$ *denote the set of regular policy-motivated equilibrium outcomes as a function of* λ *and let* $x^*(\lambda)$ *denote any element of this set.*

(i) *If* $\lambda' > \lambda \geq 0$, $q_L \notin X^*(\lambda')$*, and* $q_L \notin X^*(\lambda)$*, we must have* $U_L(x^*(\lambda'), 1) > U_L(x^*(\lambda), 1)$.
(ii) *There exists a* $\bar{\lambda}$ *such that* $\lambda \geq \bar{\lambda}$ *implies that* $X^*(\lambda) = \{q_L\}$.

Figure 1 summarizes the results of Proposition 8. When $\lambda \leq \lambda$, we find that the policy outcome (and candidate R's) position occurs at candidate R's ideal point. As we increase λ, the policy outcome moves towards that vote maximizing position y^*. When $\lambda = 0$, the policy outcome exactly equals the vote-maximizing outcome. As λ increases further, candidate L starts to win the election and the policy outcomes move towards candidate L's ideal point. Eventually, when $\lambda \geq \bar{\lambda}$, the policy outcome reaches candidate L's ideal point.[22] These comparative statics continue to hold even in the multidimensional case, and in the unlikely event that multiple equilibria are present.

The result of Proposition 8 is related to the marginality hypothesis (Fiorina 1973) which posits that electorally weak candidates are more likely to adopt centrist positions. More recently, the theoretical literature has interpreted 'weak' as 'low valence' (Groseclose 2001; Adams et al. 2005; Meirowitz 2007). Our result differs from Groseclose's main theoretical result because Groseclose considers the case where one candidate has a small advantage and because he assumes that the candidates are uncertain about the position of the median voter. Our result is consistent with Adams, Merrill, and Grofman, who instead assume that candidates are uncertain about their valence advantage.

Next, we consider a different type of comparative static. We consider changes in the distribution of voter ideal points. We consider a class of models indexed by θ

[22] One may wonder why the losing candidate does not take the median voter's position in this figure. The losing candidate will take the median voter's position in the one-dimensional Downsian model with valence, but this result does not hold more generally. The comparative statics for the losing candidate's position in the non-policy advantage are ambiguous.

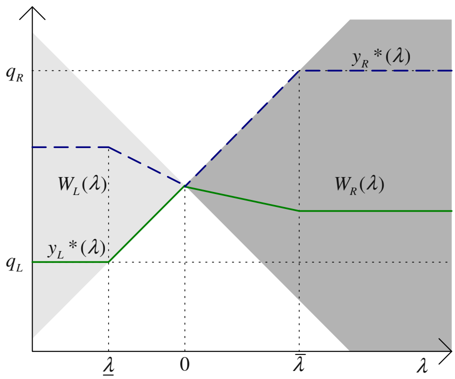

Fig. 1 Comparative statics for non-policy factors

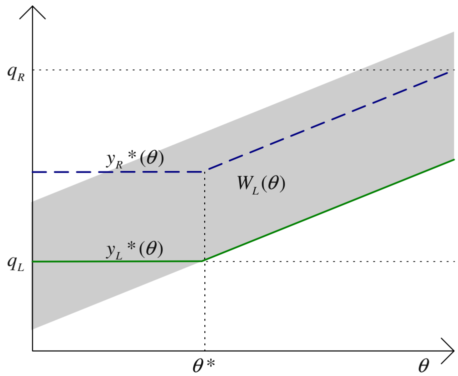

Fig. 2 Comparative statics for voter ideal points

where $f(v, z; \theta) = g(v - \theta, z)$. Suppose further that we shift the set of admissible strategies $Y(\theta)$ along with everything else. We will show that shifting the distribution by θ involves shifting the set $W_L(\theta)$ is the same way.

Proposition 9 *Consider a collection of models indexed by* θ *where* $f(v, z; \theta) = g(v-\theta, z)$. *Suppose that assumptions* (A1–A5) *and* (B1–B4) *hold for all* θ. *Suppose that the set of admissible policies* $Y(\theta)$ *changes with* θ *such that* $Y(\theta) = \{y + \theta : y \in Y(0)\}$. *Then* $W_L(\theta) = \{y + \theta : y \in W_L(0)\}$.

The consequences of this result are most easily seen in the one-dimensional case. Figure 2 gives one such example when $\tilde{z} > 0$. If the distribution of voter ideal points is shifted by θ, then the set $W_L(\theta)$ is shifted by θ as well. As long as $q_L \in W_L(\theta)$, shifting the distribution of ideal points does not effect the equilibrium. When $q_L \notin W_L(\theta)$, the policy outcome will be shifted along with the set $W_L(\theta)$.

The final form of comparative static concerns the ideal points of the candidates. Here, we index regular equilibria using the candidate's ideal points q_L and q_R. Consider the one-dimensional case and suppose that W_L has the form $W_L = [a, b]$. Then we have,

$$y_L^*(q_L, q_R) = \begin{cases} a & q_L < a \\ q_L & a \le q_L \le b \\ b & q_L > b \end{cases}$$

The result indicates that the equilibrium outcome is only affected by the position of the advantaged candidate. Furthermore, the equilibrium outcome only changes when the advantaged candidate's ideal point is contained in the set W_L. The dynamics are potentially more interesting when $J > 1$ since the shape of the candidate's utility function and the shape of W_L can matter as well.

7 Existence and the pseudo-core

In this section, we further characterize equilibrium existence and the pseudo-core. We show that if the vote share functions are initially concave, then decreasing the degree of policy voting preserves this concavity. We show that the vote share functions are concave if the weight voter's place on policy is sufficiently small. We show that a vote-maximizing point will fail to exist when the weight voter's place on policy in sufficiently large. These results can be applied to characterize equilibrium existence when neither candidate is advantaged.

We consider the class of models indexed by γ with $h(x; \gamma) = \gamma m(x)$. Here, the parameter γ controls the degree of policy voting (or inversely, the degree of non-policy voting).

Proposition 10 *Consider a class of models indexed by γ with $h(x; \gamma) = \gamma m(x)$. Suppose that assumptions* (A1–A5) *hold for all $\gamma > 0$.*

(i) *If $s_L(y_L, y_R; \gamma)$ is concave in y_L and $s_R(y_L, y_R; \gamma)$ is concave in y_R and $0 < \gamma' < \gamma$, then $s_L(y_L, y_R; \gamma')$ is concave in y_L and $s_R(y_L, y_R; \gamma')$ is concave in y_R.*

(ii) *There exists a $\gamma > 0$ sufficiently small such that $s_L(y_L, y_R; \gamma)$ is concave in y_L and $s_R(y_L, y_R; \gamma)$ is concave in y_R.*

(iii) *Suppose that,*

$$\int_{v \in \mathbb{R}^J} \frac{\partial m}{\partial x}(y^* - v) \frac{\partial m}{\partial x}(y^* - v)^T f'_{z|v}(0|v) f_v(v) \mathrm{d}v \neq 0$$

Then there exists a $\bar{\gamma} > 0$ such that for all $\gamma \geq \bar{\gamma}$, y^* is not a vote-maximizing point.[23]

Proposition 10, part (ii) indicates that the conditions of Proposition 7 will be met when γ is sufficiently small. It, therefore, follows that when $\tilde{z} = 0$, a policy-motivated equilibrium will exists when γ is sufficiently small. Proposition 10, part (iii) indicates

[23] Note that Proposition 1 implies that y^* does not depend on the parameter γ.

that a policy-motivated equilibrium will fail to exist if we are sufficiently close to the Downsian model, unless a restrictive condition is met.[24]

We next show that increasing a candidate's non-policy advantage strictly increases the size of the pseudo-core (provided it is non-empty), suggesting that increasing a candidate's non-policy advantage increases the likelihood that a policy-motivated equilibrium exists. We also show that a policy-motivated equilibrium is guaranteed to exist if one of the candidates has a sufficiently large non-policy advantage.

Proposition 11 *Consider a collection of models indexed by* λ *where* $f(v, z) = g(v, z - \lambda)$ *with* $\tilde{z}(0) = 0$. *Suppose that conditions* (A1–A5) *and* (B1–B4) *hold for all* λ. *Let* $W_L(\lambda)$ *denote the pseudo-core as a function of* λ.

- (i) *If* $\lambda' > \lambda \geq 0$, *then* $W_L(\lambda) \subset W_L(\lambda')$, *where the inclusion in strict if* $W_L(\lambda) \neq \emptyset$ *and* $W_L(\lambda) \neq Y$.
- (ii) $W_L(0) \subset \{y^*\}$
- (iii) *There exists a* $\bar{\lambda} > 0$ *such that for all* $\lambda \geq \bar{\lambda}$, $q_L \in W_L(\lambda)$ *and* (y_L, y_R, t) *is an equilibrium if and only if* $y_L = q_L$.

Proposition 5 indicated three important conditions necessary to guarantee existence when one candidate is advantaged: the boundary of W_L is differentiable, the interior of the pseudo-core is non-empty, and s_L is quasi-concave in its first argument. All three conditions are difficult to verify because s_L and W_L do not admit simple analytical expressions except for the one-dimensional Downsian model with valence. We argued that the differentiability condition would be met generically. Proposition 11 indicates that the pseudo-core is likely to be non-empty. In practical applications of the standard probabilistic voting model, it has been found that vote-maximizing points typically exist (Erikson and Romero 1990). The results of Propositions 7 and 10 indicate that this will carry over the spatial model with non-policy factors with policy-motivated candidates.

Groseclose (2007) shows that the pseudo-core in non-empty in the case where $D = 1$ and v and z are independent. This result captures the idea that adding non-policy factors into the one-dimensional Downsian model preserves the non-emptiness of the core. Proposition 11 shows that this generalizes to the multidimensional case and to the case where v and z are not independent.

8 Computational example

The results indicate that equilibria are extremely likely to exist in the one-dimensional case. Here, we consider the two-dimensional case using a computational example. We suppose that v and z are independent for simplicity. We assume that $h(x) = -\gamma x^2$ and that $z \sim N(\lambda, 1)$. We assume that v is uniformly distributed over the set $\{v \in \left[-\frac{1}{2}, \frac{1}{2}\right]^2 : v_1 \leq v_2\}$. This choice is selected because a core will fail to exist in the multidimensional Downsian model when the ideal points have such a distribution.

[24] If v and z are independent, the condition in Proposition 10, part (iii), will hold only if $f_z'(0) = 0$.

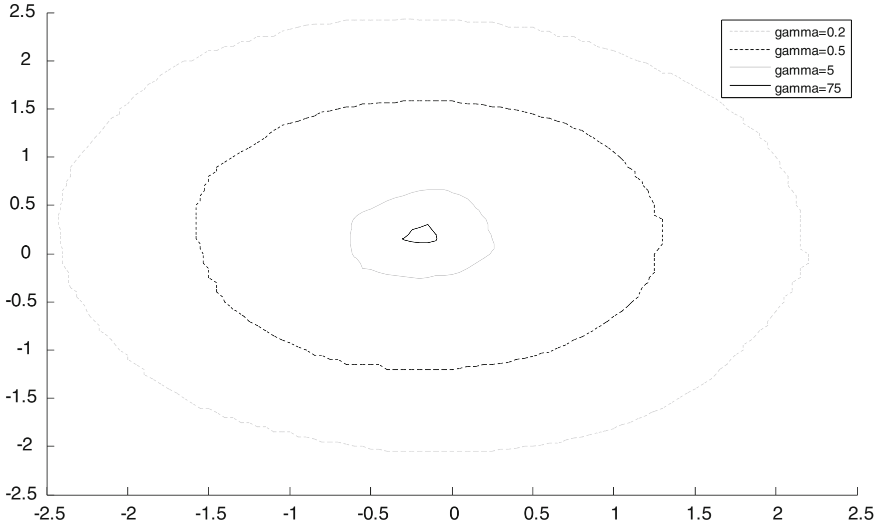

Fig. 3 The pseudo-core

We consider the case where $\lambda > 0$. The case where $\lambda = 0$ is less interesting because we know theoretically that $W_L \subset \{y*\} = \left\{\left(-\frac{1}{3}, -\frac{1}{6}\right)\right\}$.[25] For simplicity, we fix $\lambda = 1$ and vary $\gamma \in \{0.2, 0.5, 5, 75\}$. We select $Y = \left[-\frac{5}{2}, \frac{5}{2}\right]^2$. The results are shown in Fig. 3.[26] The results indicate that the pseudo-core is non-empty even for implausibly high values of γ. A high value of gamma corresponds to a high level of policy-voting relative to the non-policy advantage of candidate L. This example is pessimistic because empirical investigations of the spatial model find that voter preferences display less asymmetry than is present in the example.[27]

9 Conclusions and discussion

In this paper, we developed the theory of policy-motivated candidates in the spatial model with non-policy factors. We found that vote-maximizing equilibria typically fail to exist when one candidate has a large non-policy advantage. Contrarily, we provided

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[25] It follows from Proposition 11 that $W_L \subset \{y^*\}$. Because h is quadratic, $y^* = E[v]$. We can calculate that $E[v] = \left(-\frac{1}{3}, -\frac{1}{6}\right)$.

[26] We numerically compute the boundary of W_L by plotting the $y_L = \frac{1}{2}$ contour of $g(y_L) = \min_{y_R \in Y} s_L(y_L, y_R)$using a grid of 101 by 101 points. The integral over (v, z) is computed using $N = 100$ simulation draws. Interpolation is used to find the contour, so the result will not fall directly on the grid. We found that the calculations were not sensitive to employing a larger grid size.

[27] We note that in this example, for extremely large values of γ, the pseudo-core will be empty. In this case, our model approaches the Downsian model. For the distribution of ideal points used in the example, the Downsian model does not have a core. As we approach this case, eventually the pseudo-core must cease to exist. This result is true whenever the *core* is nonempty, but establishing this result is beyond the scope of this paper.

conditions for the existence of a policy-motivated equilibrium. We further provided a characterization of regular equilibria, and provided conditions for the existence of a unique regular equilibrium.

Our results indicated that while a strong candidate must temper his policy proposals in order to satisfy his constituents, his advantage on the non-policy dimension allows him to propose policies that deviate from the political center. He is able to move the policy outcome away from the vote-maximizing point and towards his own ideal point. The advantaged candidate will position as close to his ideal point as possible while still winning the election. This balancing of personal policy preferences and the need to secure reelection is a critical feature of elections, which we show is a general property of two candidate competition.

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Appendix

Lemmas 1–2

Lemma 1 *Suppose that assumption* (A1–A5) *hold. Then* $s_L(y_L, y_R)$ *is jointly continuous and jointly continuously differentiable in* y_L *and* y_R.

Proof Recall that, $s_L(y_L, y_R) = \int_{v\in\mathbb{R}^J}\int_{z=h(y_R-v)-h(y_L-v)}^{\infty} f(v,z)\mathrm{d}z\mathrm{d}v = \int_{v\in\mathbb{R}^J}\left\{1-F_{z|v}(h(y_R-v)-h(y_L-v)|z)\right\}f_v(v)\mathrm{d}v$ Continuity of h and $F_{z|v}$ implies that $1-F_{z|v}(h(y_R-v)-h(y_L-v)|z)$ is continuous, and hence s_L is continuous as well. Differentiability follows from the differentiability of h and $F_{z|v}$.

Lemma 2 *Let* $A\subset\mathbb{R}^J$ *be a convex set with* $x^*\in\partial A$. *Suppose that* ∂A *is a* C^1 *manifold in a neighborhood of* x^*. *Let* P *denote the supporting hyperplane of* x^* *and let* H *denote the half-space to* P *that does not contain* A. *For each* $x_1\in H$, *there exists a* $x_2\in A$ *and* $\lambda\in(0,1)$ *such that* $x^*=\lambda x_1+(1-\lambda)x_2$.

Proof Suppose this were not the case. Then the vector (x^*, x_1) is an element of the tangent space of ∂A at x^*. Hence, we have a tangent vector that is not part of P. However, since ∂A is a C^1 manifold is a neighborhood of x^*, P must contain all tangent vectors. Hence, the result is proved by contradiction.[28]

Proofs of Propositions

Proof of Proposition 1 (i) Taking first-order conditions leads to the following necessary conditions for an interior vote-maximizing point,

[28] The proof of this result can easily be visualized by considering the case where $A\subset\mathbb{R}^2$.

$$\int_{v\in\mathbb{R}^J} \frac{\partial h}{\partial x_j}(y_L - v) f\,(v, h(y_R - v) - h(y_L - v))\,\mathrm{d}v = 0 \quad \text{for } j = 1, \ldots, J$$

$$\int_{v\in\mathbb{R}^J} \frac{\partial h}{\partial x_j}(y_R - v) f\,(v, h(y_R - v) - h(y_L - v))\,\mathrm{d}v = 0 \quad \text{for} \quad j = 1, \ldots, J$$

we assume that there exists an interior vote-maximizing point with $y_L^* \neq y_R^*$. The two necessary conditions above imply that,

$$\int_{v\in\mathbb{R}^J} \left[\frac{\partial h}{\partial x_j}(y_R^* - v) - \frac{\partial h}{\partial x_j}(y_L^* - v)\right] f\left(v, h(y_R^* - v) - h(y_L^* - v)\right)\mathrm{d}v$$
$$= 0 \quad \text{for} \quad j = 1, \ldots, J$$

This implies that,

$$x^T \int_{v\in\mathbb{R}^J} \left[\frac{\partial h}{\partial x}(y_R^* - v) - \frac{\partial h}{\partial x}(y_L^* - v)\right] \times f\left(v, h(y_R^* - v) - h(y_L^* - v)\right)\mathrm{d}v = 0$$

for all $x \in \mathbb{R}^J$. Taking a first-order Taylor expansion of $\frac{\partial h}{\partial x}(y_R^* - v)$ around $y_L^* - v$ yields $x^T\left[\frac{\partial h}{\partial x}(y_R^* - v) - \frac{\partial h}{\partial x}(y_L^* - v)\right] = x^T H(y_R^* - y_L^*)$ where H is equal to $\frac{\partial^2 h}{\partial x \partial x^T} h(a)$ and a are the mean values from the Taylor expansion. Setting $x = y_R^* - y_L^*$, Assumption (A3) implies that $x^T\left[\frac{\partial h}{\partial x}(y_R^* - v) - \frac{\partial h}{\partial x}(y_L^* - v)\right] = x^T H x < 0$. Since this holds for all v, we have that,

$$x^T \int_{v\in\mathbb{R}^J} \left[\frac{\partial h}{\partial x}(y_R^* - v) - \frac{\partial h}{\partial x}(y_L^* - v)\right] \times f\left(v, h(y_R^* - v) - h(y_L^* - v)\right)\mathrm{d}v < 0$$

which is a contradiction. Hence, any vote-maximizing point must satisfy $y_L^* = y_R^* = y^*$.

(ii) Plugging in $y_L^* = y_R^* = y^*$ to the first-order conditions gives,

$$\int_{v\in\mathbb{R}^J} \left(\frac{\partial h}{\partial x_j}(y^* - v)\right) f(v, 0)\mathrm{d}v = 0 \quad \text{for} \quad j = 1, \ldots, J$$

immediately proving the result.

Proof of Proposition 2 (i) We prove this result by contradiction. Suppose that there exist two interior vote-maximizing points y^* and y with $y \neq y^*$. In this case, we have $\int_{v\in\mathbb{R}^J} \frac{\partial h}{\partial x}(y^* - v) f_{v|z}(v|0)\mathrm{d}v = 0$ and $\int_{v\in\mathbb{R}^J} \frac{\partial h}{\partial x}(y - v) f_{v|z}(v|0)\mathrm{d}v = 0$. For each v, we take a second-order Taylor expansion of $\frac{\partial h}{\partial x}(y^* - v)$ around $y - v$ and obtain

$\frac{\partial h}{\partial x}(y^* - v) = \frac{\partial h}{\partial x}(y - v) + H(v)(y^* - y)$ where $H(v)$ is a negative definite matrix. Multiplying both sides by $f_{v|z}(v|0)$ and integrating over $v \in \mathbb{R}^J$, we obtain,

$$\int\limits_{v\in\mathbb{R}^J} \frac{\partial h}{\partial x}(y^* - v) f_{v|z}(v|0)\mathrm{d}v = \int\limits_{v\in\mathbb{R}^J} \frac{\partial h}{\partial x}(y - v) f_{v|z}(v|0)\mathrm{d}v + \int\limits_{v\in\mathbb{R}^J} H(v)(y^* - y) f_{v|z}(v|0)\mathrm{d}v$$

Hence, we have $\int_{v\in\mathbb{R}^J} H(v)(y^* - y) f_{v|z}(v|0)\mathrm{d}v = 0$. Multiplying both sides by $(y^* - y)^T$, we obtain,

$$\int\limits_{v\in\mathbb{R}^J} (y^* - y)^T H(v)(y^* - y) f_{v|z}(v|0)\mathrm{d}v = 0$$

Since $H(v)$ is negative definite by assumption and $f_{v|z}(v|0) > 0$ for all $v \in \mathbb{R}^J$, we have $(y^* - y)^T H(v)(y^* - y) f_{v|z}(v|0) < 0$ for all $v \in \mathbb{R}^J$, which implies that,

$$\int\limits_{v\in\mathbb{R}^J} (y^* - y)^T H(v)(y^* - y) f_{v|z}(v|0)\mathrm{d}v < 0$$

Hence, the result holds by contradiction.

(ii) In Lemma 1, we show that the vote-share functions are continuous. Hence, the result is a direct consequence of the Debreu–Fan–Glicksberg theorem (Debreu 1952; Fan 1952; Glicksberg 1952), discussed in Duggan (2005).

Proof of Proposition 3 Suppose that (y_L^*, y_R^*, t^*) satisfies $s_L(y_L^*, y_R^*) < \frac{1}{2}$. Then candidate L can clearly improve his utility by moving to y_R^* since this will lead to the same policy outcome, but will increase his probability of holding office (since L wins with probability 1 when both candidates locate in the same place). Alternatively, suppose that $s_L(y_L^*, y_R^*) > \frac{1}{2}$ and $y_L^* \neq q_L$. Since $s_L(y_L, y_R)$ is continuous in y_L, there exists an $\varepsilon_1 > 0$ small enough such that $s_L(y_L^* + \varepsilon_1 p, y_R^*) > \frac{1}{2}$ for any direction $p_1 \in \mathbb{R}^J$ with $\|p_1\| = 1$. Furthermore, since $y_L^* \neq q_L$ and $U_L(x, 1)$ is strictly concave, y_L^* is not a local maximum of $U_L(x, 1)$. Hence, there exist a $p_2 \in \mathbb{R}^J$ with $\|p_2\| = 1$ such that $U_L(y_L^* + \varepsilon_2 p_2, 1) > U_L(y_L^*, 1)$ for $\varepsilon_2 > 0$ small enough. Picking $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, we can see that candidate L can improve his utility by moving to $y_L^* + \varepsilon p_2$.[29] Hence, any equilibrium with $y_L^* \neq q_L$ must satisfy $s_L(y_L^*, y_R^*) = \frac{1}{2}$.

We can rule out the situation in which L wins with probability 0 since candidate L could improve his utility by moving to y_R^*. Now consider an interior non-satiated equilibrium with $0 < t^* < 1$. If $y_L^* = y_R^* = y^*$, then $s_L(y_L^*, y_R^*) > \frac{1}{2}$. This implies that (y_L^*, y_R^*) is not a local vote-maximizing point for both candidate. Since (y_L^*, y_R^*)

[29] We will invoke this argument many times in the proofs below. We will not repeat all the details when we invoke this argument later.

is a non-satiated equilibrium by assumption, at least one candidate can improve his utility by making a sufficiently small move (by the above argument). Hence, we cannot have $0 < t^* < 1$ in a non-satiated equilibrium.

Proof of Proposition 4 Consider any equilibrium with $y_L^* \in W_L$. Clearly, $y_L^* \in \arg\max_{y_L \in W_L} U_L(y_L, 1)$ is necessary for equilibrium to hold, since otherwise, candidate L could improve his utility by moving to an element of $\arg\max_{y_L \in W_L} U_L(y_L, 1)$. Now suppose that $y_L^* \neq q_L$. We know that $s_L(y_L^*, y_R^*) = \frac{1}{2}$ from Proposition 3. We also know that $s_R(y_L^*, y_R) \geq \frac{1}{2}$ by the definition of W_L, so $s_L(y_L^*, y_R^*) = \frac{1}{2}$ immediately implies that have $y_R^* \in \arg\max_{y_R \in Y} s_R(y_L^*, y_R)$.

Proof of Proposition 5 For the existence part of the proof, we will show that there exists a regular equilibrium (y_L^*, y_R^*, t^*) with $t^* = 1$. Consider first the case where $y_L^* = q_L$. Since candidate L wins with probability 1, it follows that candidate L cannot improve his utility by moving. Furthermore, $y_L^* \in W_L$ implies that $s_R(y_L^*, y_R) \geq \frac{1}{2}$ for all $y_R \in Y$, so candidate R cannot improve his utility by moving.

Now consider the case where $y_L^* \neq q_L$. We first show that there exists a point $(y_L^*, y_R^*) \in Y^2$ with $y_L^* \in \arg\max_{y_L \in W_L} U_L(y_L, 1)$ and $y_R^* \in \arg\max_{y_R \in Y} s_R(y_L^*, y_R)$. We then show that such a point will be an equilibrium. We first want to show that $\arg\max_{y_L \in W_L} U_L(y_L, 1)$ is nonempty. We need to show that the set $W_L = \{y_L \in Y : \min_{y_R \in Y} s_L(y_L, y_R) \geq \frac{1}{2}\}$ is compact. Since $W_L \subset Y$, this set must be bounded. By the compactness of Y and the continuity of s_L in both its arguments, the theorem of the maximum tells us that W_L is compact as well. Furthermore, W_L is non-empty by assumption. Now, since U_L is continuous, this implies that U_L must attain a maximum over W_L. Next, we must show that $\arg\max_{y_R \in Y} s_R(y_L^*, y_R)$ is non-empty. Compactness of Y along with the continuity of s_R implies that this must be the case.

Now, to verify that (y_L^*, y_R^*, t^*) is an equilibrium when $t^* = 1$, we must demonstrate that neither candidate can improve his utility by deviating. It is clear that candidate R cannot improve his utility by moving, since no matter where he locates, candidate L will win the election with probability 1. Since candidate R cannot change who wins the elections, he cannot change the policy outcome either.

The final step is to show that candidate L cannot improve his utility by moving. We can show that a point $(y_L^*, y_R^*) \in Y^2$ satisfying $y_L^* \in \arg\max_{y_L \in W_L} U_L(y_L, 1)$ and $y_R^* \in \arg\max_{y_R \in Y} S_R(y_L^*, y_R)$ must also satisfy $S_L(y_L^*, y_R^*) = \frac{1}{2}$. Notice that $y_L^* \in \arg\max_{y_L \in Y} U_L(y_L, 1)$ and concavity of $U_L(y_L, 1)$ implies that $y_L^* \in \partial W_L$. If $S_L(y_L^*, y_R^*)$, then $y_L^* \notin W_L$. If $s_L(y_L^*, y_R^*) > \frac{1}{2}$, then continuity of $\arg\max_{y_R \in Y} s_R(y_L, y_R)$ over y_L implies that $s_L(y_L, y_R) > \frac{1}{2}$ for y_L is a neighborhood of y_L^* and $y_R \in \arg\max_{y_R \in Y} s_R(y_L, y_R)$. This tells us that $y_L \in W_L$ in a neighborhood of y_L^*, which contradicts $y_L^* \in \partial W_L$. Hence, $s_L(y_L^*, y_R^*) = \frac{1}{2}$ must hold.

Now, let P denote the supporting hyperplane of ∂W_L at y_L^*. Let H denote the half-space to P which does not contain W_L. Suppose that $y_R^* \in H$. By Lemma 2, there exists a $y_L \in \text{int}(W_L)$ and $\lambda \in (0, 1)$ such that $y_L^* = \lambda y_R^* + (1 - \lambda) y_L$. Notice that, $s_L(y_R^*, y_R^*) > \frac{1}{2}$, $s_L(y_L^*, y_R^*) = \frac{1}{2}$, and $s_L(y_L, y_R^*) \geq \frac{1}{2}$. This contradicts quasi-concavity however, which implies that $y_R^* \notin H$, which further implies that $U_L(y_R^*, 1) < U_L(y_L^*, 1)$. This step also implies that the equilibrium exhibits divergence, as stated in the proposition. Now, since $U_L(y_R^*, 1) > U_L(y_R^*, 0)$, candidate L cannot improve his utility by moving to a location where he looses the election.

We also have to show that candidate L cannot improve his utility by moving to a location where he wins. Consider any $y_L \in Y$ such that $U_L(y_L, 1) > U_L(y_L^*, 1)$ and $s_L(y_L, y_R^*) \geq \frac{1}{2}$. By construction, we must have $y_L \in H$. Furthermore, differentiability of U_L implies that ∂U_L is a C^1 manifold. By Lemma 2, there exists a $y_L' \in W_L$ and $\lambda \in (0, 1)$ such that $y_L^* = \lambda y_L + (1 - \lambda)y_L'$. Since $s_L(y_L, y_R^*) \geq \frac{1}{2}$ and $s_L(y_L', y_R^*) \geq \frac{1}{2}$ by construction, quasi-concavity implies that $s_L(y_L^*, y_R^*) > \frac{1}{2}$, which is a contradiction. Hence, candidate L cannot improve his utility, proving that the candidate equilibrium is indeed an equilibrium.

The next step of the proof is to show that the position of candidate L is uniquely determined in a regular equilibrium. Since U_L is convex in its first argument by assumption, convexity of W_L will imply that the solution to $y_L^* \in \arg\max_{y_L \in W_L} U_L(y_L, 1)$ is unique. It is thus sufficient to show that W_L is convex. Consider any two point $y_1, y_2 \in W_L$. These points satisfy $\min_{y_R \in Y} s_L(y_1, y_R) \geq \frac{1}{2}$ and $\min_{y_R \in Y} s_L(y_2, y_R) \geq \frac{1}{2}$. Now consider a point $y_L = \lambda y_1 + (1 - \lambda)y_2$. We want to show that $y_L \in W_L$. Since,

$$\begin{aligned}\min_{y_R \in Y} s_L(y_L, y_R) &= \min_{y_R \in Y} s_L(\lambda y_1 + (1 - \lambda)y_2, y_R)\\ &\geq \min_{y_R \in Y} \{\min\{s_L(y_1, y_R), s_L(y_2, y_R)\}\}\end{aligned}$$

by the quasi-concavity of s_L in the first argument. Next,

$$\min_{y_R \in Y} \{\min\{s_L(y_1, y_R), s_L(y_2, y_R)\}\} = \min\{\min_{y_R \in Y} s_L(y_1, y_R), \min_{y_R \in Y} s_L(y_2, y_R)\} \geq \frac{1}{2}$$

which implies that W_L is indeed convex.

Next, we want to show that if s_L is quasi-concave in its second argument, then $\text{int}(W_L) \neq \emptyset$. By the properties of s_L and s_R, we have that a vote-maximizing equilibrium exists. Hence, $s_L(y^*, y^*) \leq s_L(y^*, y_R)$ for all $y_R \in Y$. Since $s_L(y^*, y^*) > \frac{1}{2}$, we have that $y^* \in W_L$ and W_L is non-empty. Now, consider a point $y_L = y^* + p\varepsilon$ for $||p|| = 1$ and $\varepsilon > 0$. Compactness of Y and continuity of s_L implies that s_L is uniformly continuous in both arguments over Y^2. Since $s_L(y^*, y_R) \geq s_L(y^*, y^*) > 0$ for all $y_R \in Y$, uniform continuity implies that $s_L(y^* + p\varepsilon, y_R) > 0$ for $\varepsilon > 0$ small enough. Hence, there exists a neighborhood around y^* which is contained in W_L, and consequently, $\text{int}(W_L) \neq \emptyset$.

Finally, we want to prove that if s_L is quasi-concave in its second argument, then an interior non-satiated irregular equilibrium does not exist. Consider a potential equilibrium (y_L^*, y_R^*) with $y_L^* \neq q_L$, $y_R^* \neq q_R$, and $y_L^* \notin W_L$. By proposition 4, we must have $s_L(y_L^*, y_R^*) = \frac{1}{2}$. Since $y_L^* \notin W_L$, there exists a $y_R \in Y$ such that $s_L(y_L^*, y_R) < \frac{1}{2}$. Define $y_R' = \lambda y_R^* + (1 - \lambda)y_R$ where $\lambda \in (0, 1)$. Then, by quasi-convexity, we have $s_L(y_L^*, y_R') < \frac{1}{2}$. By selecting λ small enough, continuity of U_R implies that we can find a point y'_R where $U_R(y'_R, 1) > U_R(y^*_R, 0)$. Hence, (y^*_L, y^*_R) cannot be a policy-motivated equilibrium.

Proof of Proposition 6 Consider an interior equilibrium with $y^*_L \neq q_L$ and $y^*_R \neq q_R$. We first show that $s_L(y^*_L, y^*_R) = \frac{1}{2}$ must hold in equilibrium. If $t^* \in (0, 1)$ and $s_L(y^*_L, y^*_R) \neq \frac{1}{2}$, then the loosing candidate can clearly improve his utility by moving to the other candidate's position. If $t^* = 1$ and $s_L(y^*_L, y^*_R) < \frac{1}{2}$, then candidate L can improve his utility by moving to y^*_R. If $s_L(y^*_L, y^*_R) > \frac{1}{2}$, then candidate L can improve his utility by making a small move towards his ideal point. For a sufficiently small move, this will not affect the probability that he wins the election, but will increase his policy utility. Hence, we must have $s_L(y^*_L, y^*_R) = \frac{1}{2}$ when $t^* = 1$. The case where $t^* = 0$ can be proved analogously.

Now suppose that $t^* \in (0, 1)$ and $y^*_L = y^*_R = y^*$ fails to hold. At least one candidate is not at a local maximum. This candidate can improve his utility by making a sufficiently small move that increases his probability of winning to 1. Hence, when $t^* \in (0, 1)$, all equilibria of this form satisfy $y^*_L = y^*_R = y^*$.

Proof of Proposition 7 First, we will show that (y^*, y^*, t^*) is an equilibrium for all $t^* \in [0, 1]$. By Proposition 2, (y^*, y^*) is a vote-maximizing point. It is immediately clear that neither candidate can change the policy outcome by moving, and neither candidate can increase the probability with which they win the election by moving. Hence, (y^*, y^*, t^*) is a policy-motivated equilibrium. By Proposition 2, we know that there is only one interior vote-maximizing point.

The remaining part of the proof shows that there does not exist any other type of equilibrium. If $t^* \in (0, 1)$, Proposition 6 tells us there is nothing to show. Consider an interior non-satiated equilibrium, (y^*_L, y^*_R, t^*). Suppose first that $s_L(y^*_L, y^*_R) = \frac{1}{2}$. If $y^*_L = y^*$ and $y^*_R = y^*$, there is nothing to show. Suppose instead that $y^*_R \neq y^*$. We have $s_L(y^*, y^*_R) = \frac{1}{2}$ and $s_L(y^*, y^*) = \frac{1}{2}$, which implies that $s_L(y^*, \lambda y^*_R + (1-\lambda)y^*) < \frac{1}{2}$ for $\lambda \in (0, 1)$, by quasi-convexity. By selecting λ small enough, candidate R can improve his utility. Hence, this cannot be an equilibrium. Now assume that $y^*_L \neq y^*$. In this case, $s_L(y^*_L, y^*) \leq \frac{1}{2}$ and $s_L(y^*_L, y^*_R) = \frac{1}{2}$ implies that $s_L(y^*_L, \lambda y^*_R + (1-\lambda)y^*) < \frac{1}{2}$, by quasi-convexity. By selecting λ small enough, candidate R can improve his utility since this will allow him to win the election with probability 1, but will have little effect on his policy utility. A similar argument holds when $t^* = 0$. Hence, there does not exist another equilibrium with $s_L(y^*_L, y^*_R) = \frac{1}{2}$.

It remains to show that there does not exist an equilibrium with $s_L(y^*_L, y^*_R) \neq \frac{1}{2}$. By Proposition 6, such an equilibrium must be a satiated equilibrium, so suppose, without loss of generality, that $y^*_L = q_L$. If $s_L(y^*_L, y^*_R) < \frac{1}{2}$, then candidate L can improve his utility my moving to y^*_R. If $s_L(y^*_L, y^*_R) > \frac{1}{2}$ and $t^* < 1$, then candidate R can improve his utility by moving to y^*_L. If $s_L(y^*_L, y^*_R) > \frac{1}{2}$ and $t^* = 1$, notice that $s_L(y^*_L, y^*_L) = \frac{1}{2}$, $s_L(y^*_L, y^*) \leq \frac{1}{2}$, and $s_L(y^*_L, \lambda y^* + (1-\lambda)y^*_L) < \frac{1}{2}$. Selecting λ small enough allows candidate R to improve his utility by winning the election with probability one, but barely moving the policy outcome, proving the result.

Proof of Proposition 8 (i) Notice that,

$$s_L(y_L, y_R; \lambda') = \int\limits_{v \in \mathbb{R}^J} \int\limits_{z=h(y_R-v)-h(y_L-v)}^{\infty} g(v, z-\lambda')\mathrm{d}z\mathrm{d}v$$
$$= \int\limits_{v \in \mathbb{R}^J} \int\limits_{z=h(y_R-v)-h(y_L-v)-\lambda'}^{\infty} g(v, z)\mathrm{d}z\mathrm{d}v$$

Since,

$$\left\{(v, z) : v \in \mathbb{R}^J, z > h(y_R - v) - h(y_L - v) - \lambda\right\}$$
$$\subset \left\{(v, z) : v \in \mathbb{R}^J, z > h(y_R - v) - h(y_L - v) - \lambda'\right\}$$

we have that $s_L(y_L, y_R; \lambda') > s_L(y_L, y_R; \lambda)$ for all $(y_L, y_R) \in Y^2$. Hence,

$$s_L\left(x^*(\lambda), y'_R; \lambda'\right) > s_L\left(x^*(\lambda), y_R; \lambda\right) = \frac{1}{2}$$

We must also have that $s_L(x^*(\lambda), y'_R; \lambda') \leq s_L(x^*(\lambda), y_R)$ for all y_R. Consider a move by candidate L to $x^*(\lambda) + \varepsilon p$. Since $s_L(x^*(\lambda), y_R; \lambda') > \frac{1}{2}$, continuity of s_L implies that there must exist an $\varepsilon > 0$ small enough such that $s_L\left(x^*(\lambda) + \varepsilon p, y_R; \lambda'\right) > \frac{1}{2}$. Assumption (B2) and (B3) imply that there must exist an p such that $U_L(x^*(\lambda) + \varepsilon p, 1) > U_L(x^*(\lambda), 1)$. Since candidate L would win the election with probability by positioning at $x^*(\lambda) + \varepsilon p$, no matter where candidate R locates, we must have $U_L(x^*(\lambda'), 1) > U_L(x(\lambda), 1)$.
(ii) By making λ sufficiently large, we can make $\left\{(v, z) : v \in \mathbb{R}^J, z > h(y_R - v) - h(y_L - v) - \lambda\right\}$ sufficiently large. Hence, we can make $s_L(y_L, y_R; \lambda)$ sufficiently large for any $(y_L, y_R) \in Y^2$. Moreover, since s_L is continuous and Y is a compact set, we can make $s_L(y_L, y_R; \lambda)$ arbitrarily large for each y_L, uniformly over y_R. Hence, we can ensure that W_L is arbitrarily large, indicating that we can find a $\bar{\lambda}$ such that $\lambda \geq \bar{\lambda}$ implies that $x^*(\lambda) = q_L$.

Proof of Proposition 9 Consider the change of variables $v = u + \theta$. Then we have,

$$s_L(y_L, y_R; \theta) = \int\limits_{v \in \mathbb{R}^J} \int\limits_{z=(y_L-v)^2-(y_R-v)^2}^{\infty} g(v-\theta) f_z(z)\mathrm{d}z\mathrm{d}v$$
$$= \int\limits_{u \in \mathbb{R}^J} \int\limits_{z=(y_L-\theta-u)^2-(y_R-\theta-u)^2}^{\infty} g(u) f_z(z)\mathrm{d}z\mathrm{d}u = s_L\left(y_L - \theta, y_R - \theta; \theta\right)$$

Hence,

$$\begin{aligned} W_L(\theta) &= \left\{ y_L \in Y(\theta) : \min_{y_R \in Y(\theta)} s_L(y_L, y_R; \theta) \geq \frac{1}{2} \right\} \\ &= \left\{ y_L \in Y(\theta) : \min_{y_R \in Y(\theta)} s_L\left(y_L - \theta, y_R - \theta; 0\right) \geq \frac{1}{2} \right\} \\ &= \left\{ y'_L + \theta \in Y(\theta) + \theta : \min_{y'_R + \theta \in Y(\theta)+\theta} s_L\left(y'_L, y'_R; 0\right) \geq \frac{1}{2} \right\} \\ &= \{ y + \theta : y \in W_L \} \end{aligned}$$

Proof of Proposition 10 (i) We start by characterizing the Hessians of s_L and s_R,

$$\begin{aligned} H_L(y_L, y_R; \gamma) &= \gamma[A_1(y_L, y_R) - \gamma A_2(y_L, y_R)] \\ H_R(y_L, y_R; \gamma) &= \gamma[A_1(y_L, y_R) + \gamma A_2(y_L, y_R)] \end{aligned}$$

where,

$$\begin{aligned} A_1(y_L, y_R) &= \int\limits_{v \in \mathbb{R}^J} \frac{\partial m^2}{\partial x \partial x^T}(y_L - v) f(v, m(y_R - v) - m(y_L - v)) \mathrm{d}v \\ A_2(y_L, y_R) &= \int\limits_{v \in \mathbb{R}^J} \frac{\partial m}{\partial x}(y_L - v) \frac{\partial m}{\partial x}(y_L - v)^T f_2\left(v, m(y_R - v) - m(y_L - v)\right) \mathrm{d}v \end{aligned}$$

We have that,

$$x^T H_L(y_L, y_R; \gamma) x < 0, \quad x^T H_R(y_L, y_R; \gamma) x < 0$$

for all $(y_L, y_R) \in Y^2$. Equivalently, we can write,

$$\begin{aligned} x^T A_1(y_L, y_R) x - \gamma x^T A_2(y_L, y_R) x < 0 \\ x^T A_1(y_L, y_R) x + \gamma x^T A_2(y_L, y_R) x < 0 \end{aligned}$$

Since $x^T A_1(y_L, y_R) x < 0$, we have that,

$$\gamma |x^T A_2(y_L, y_R) x| < -x^T A_1(y_L, y_R) x$$

Since $\gamma' < \gamma$, if follows that,

$$\gamma' |x^T A_2(y_L, y_R) x| < -x^T A_1(y_L, y_R) x$$

Using this, we have,

$$x^T H_L(y_L, y_R; \gamma')x = \gamma' \left[x^T A_1(y_L, y_R)x - \gamma' x^T A_2(y_L, y_R)x\right]$$
$$\leq \gamma' \left[x^T A_1(y_L, y_R)x + \gamma' |x^T A_2(y_L, y_R)x|\right]$$
$$< \gamma' \left[x^T A_1(y_L, y_R)x - x^T A_1(y_L, y_R)x\right] = 0$$

A similar result implies that,

$$x^T H_R(y_L, y_R; \gamma')x < 0$$

proving the result.
(ii) Recall that,

$$H_L(y_L, y_R; \gamma) = \gamma \left[A_1(y_L, y_R) - \gamma A_2(y_L, y_R)\right]$$
$$H_R(y_L, y_R; \gamma) = \gamma \left[A_1(y_L, y_R) + \gamma A_2(y_L, y_R)\right]$$

We first would like to show that there exists a $\delta > 0$ such that $x^T A_1(y_L, y_R)x \leq -\delta$ for all $x \neq 0$. Continuity of $A_1(y_L, y_R)$ follows from the continuity of $\frac{\partial m^2}{\partial x \partial x^T}$, f, and m. Define,

$$q(y_L, y_R) = \min_{(y_L, y_R) \in Y^2} x^T A_1(y_L, y_R)x$$

The minimum is well defined because $A_1(y_L, y_R)$ is continuous and Y^2 is compact. Since $\frac{\partial m^2}{\partial x \partial x^T}$ is negative definite and f is positive, it follows that $x^T A_1(y_L, y_R)x < 0$ for all $(y_L, y_R) \in Y^2$. Hence, it follows that there exists a $\delta > 0$ such that $q(y_L, y_R) = -\delta$. Hence, there exists a $\delta > 0$ such that $x^T A_1(y_L, y_R)x \leq -\delta$ for all $(y_L, y_R) \in Y^2$. Next, we would like to show that $A_2(y_L, y_R)$ is bounded over $(y_L, y_R) \in Y^2$. Since $\frac{\partial m}{\partial x}$ and f_2 are continuous, it follows that $A_2(y_L, y_R)$ is continuous as well. Since Y^2 is compact, it follows that there exists a $B < \infty$ such that $||A_2(y_L, y_R)||_F \leq B$ for $(y_L, y_R) \in Y^2$.

Now, consider any x with $||x||_2 = 1$.[30] We have that,

$$x^T A_1(y_L, y_R)x \leq -\delta \text{ for all } (y_L, y_R) \in Y^2$$

It follows that,

$$|x^T A_2(y_L, y_R)x| \leq ||x||_2||A_2(y_L, y_R)x||_2 \leq ||x||_2^2||A_2(y_L, y_R)||_F$$
$$= ||A_2(y_L, y_R)||_F \leq B$$

[30] Here, $||.||_2$ denotes the L2 norm for vectors and $||.||_F$ denotes the Frobenius norm for matrices. Recall that $|x'y| \leq ||x||_2||y||_2$ (the Cauchy–Schwarz inequality) and $||Ax||_F \leq ||A||_F||x||_2$.

for all $(y_L, y_R) \in Y^2$. Combining these, we have,

$$x^T H_L(y_L, y_R; \gamma)x \leq -\gamma[\delta + \gamma B] \quad \text{for all} \quad (y_L, y_R) \in Y^2$$
$$x^T H_R(y_L, y_R; \gamma)x \leq -\gamma[\delta + \gamma B] \quad \text{for all} \quad (y_L, y_R) \in Y^2$$

Selecting $\gamma = \frac{\delta}{B}$, we have,

$$x^T H_L(y_L, y_R; \gamma)x \leq -\frac{2\delta^2}{B} < 0 \quad \text{for all} \quad (y_L, y_R) \in Y^2$$
$$x^T H_R(y_L, y_R; \gamma)x \leq -\frac{2\delta^2}{B} < 0 \quad \text{for all} \quad (y_L, y_R) \in Y^2$$

Now, consider any $w \neq 0$. Selecting $x = \frac{w}{||w||_2}$, we have,

$$\left(\frac{w}{||w||_2}\right)^T H_L\left(y_L, y_R; \frac{\delta}{B}\right)\left(\frac{w}{||w||_2}\right) < 0,$$
$$\left(\frac{w}{||w||_2}\right)^T H_R\left(y_L, y_R; \frac{\delta}{B}\right)\left(\frac{w}{||w||_2}\right) < 0$$

for all $(y_L, y_R) \in Y^2$. Multiplying both sides by $||w||_2^2$, we have that $H_L\left(y_L, y_R; \frac{\delta}{B}\right)$ and $H_R\left(y_L, y_R; \frac{\delta}{B}\right)$ are negative definite, proving that $s_L\left(y_L, y_R; \frac{\delta}{B}\right)$ is concave in its first argument and $s_R\left(y_L, y_R; \frac{\delta}{B}\right)$ is concave in its second argument.
(iii) Consider the second order conditions evaluated at $y_L = y_R = y^*$. By assumption, there exists an $x \neq 0$ such that $x^T A_2(y^*, y^*)x < 0$. Define,

$$\bar{\gamma} = 2\frac{x^T A_1(y^*, y^*)x}{x^T A_2(y^*, y^*)x} > 0$$

For all $\gamma \geq \bar{\gamma}$, we have,

$$x^T H_L(y^*, y^*; \gamma)x = \gamma\left[x^T A_1(y_L, y_R)x - \gamma x^T A_2(y_L, y_R)x\right] >$$
$$\gamma\left[x^T A_1(y_L, y_R)x - \bar{\gamma} x^T A_2(y_L, y_R)x\right] = -\gamma x^T A_1(y^*, y^*)x > 0$$

Alternatively, there must exists an $x \neq 0$ such that,

$$x^T A_2(y^*, y^*)x > 0$$

Selecting,

$$\bar{\gamma} = -2\frac{x^T A_1(y^*, y^*)x}{x^T A_2(y^*, y^*)x} > 0$$

for all $\gamma \geq \bar{\gamma}$, we have,

$$x^T H_R(y^*, y^*; \gamma)x = -\gamma \left[x^T A_1(y^*, y^*)x\right] > 0$$

indicating that the second-order conditions are violated for candidate R. Hence, in either case, we can select a γ large enough such that y^* is not a local vote-maximizing point.

Proof of Proposition 11 Consider $\lambda' > \lambda \geq 0$, we have that,

$$s_L\left(y_L, y_R; \lambda'\right) = \int\limits_{v \in \mathbb{R}^J} \int\limits_{z=-\infty}^{h(y_R - v) - h(y_L - v)} g\left(v, z - \lambda'\right) \mathrm{d}z\mathrm{d}v$$
$$= \int\limits_{v \in \mathbb{R}^J} G_{z|v}\left(h(y_R - v) - h(y_L - v) - \lambda'|v\right) g_v(v)\mathrm{d}v$$

Now, $G_{z|v}(z|v)$ is strictly increasing by assumption because $g(v, z)$ is a density with full support (see Assumption A4), so it follows that,

$$G_{z|v}(h(y_R - v) - h(y_L - v) - \lambda'|v) > G_{z|v}\left(h(y_R - v) - h(y_L - v) - \lambda|v\right)$$

for all $v \in \mathbb{R}^J$ and $(y_L, y_R) \in Y^2$. It therefore follows that,

$$\int\limits_{v \in \mathbb{R}^J} G_{z|v}\left(h(y_R - v) - h(y_L - v) - \lambda'|v\right) g_v(v)\mathrm{d}v$$
$$> \int\limits_{v \in \mathbb{R}^J} G_{z|v}\left(h(y_R - v) - h(y_L - v) - \lambda|v\right) g_v(v)\mathrm{d}v$$

or that,

$$s_L\left(y_L, y_R; \lambda'\right) > s_L\left(y_L, y_R; \lambda\right)$$

for all $(y_L, y_R) \in Y^2$. Since Y^2 is compact and $s_L(y_L, y_R; \lambda)$ is continuous in the first two arguments, it follows that there exists a $\delta > 0$ such that,

$$s_L\left(y_L, y_R; \lambda'\right) \geq s_L(y_L, y_R; \lambda) + \delta$$

for all $(y_L, y_R) \in Y^2$. We therefore have that,

$$s_L(y_L, y_R; \lambda) \geq \frac{1}{2} \Rightarrow s_L\left(y_L, y_R; \lambda'\right) \geq \frac{1}{2} + \delta \quad \text{for all} \quad (y_L, y_R) \in Y^2$$

It further follows that,

$$\min_{y_R \in Y} s_L(y_L, y_R; \lambda) \geq \frac{1}{2} \Rightarrow \min_{y_R \in Y} s_L\left(y_L, y_R; \lambda'\right) \geq \frac{1}{2} + \delta \quad \text{for all} \quad y_L \in Y$$

From here, it follows that $y_L \in W_L(\lambda) \Rightarrow y_L \in W_L(\lambda')$.

Now, suppose that $W_L(\lambda) \neq \emptyset$. The theorem of the maximum tells us that $q(y_L; \lambda) = \min_{y_R \in Y} s_L(y_L, y_R; \lambda)$ is continuous in y_L. Consider any $y_L \in \partial W_L$ (we showed in the proof of Proposition 5 that W_L is compact and therefore closed). We want to show that for all $\varepsilon > 0$, there exists a $y'_L \in Y - W_L$ such that $||y'_L - y_L||_2 < \varepsilon$. Since $y_L \in \partial W_L$, there must exist such a point. By the continuity of $q(y_L; \lambda)$ in y_L, it follows that there exists an $\varepsilon > 0$ small enough such that $q(y'_L; \lambda) = \frac{1}{2} + \delta$. Hence, if follows that there exists a $y'_L \notin W_L(\lambda)$ with $y'_L \in W_L(\lambda')$, proving the result.

(ii) Consider any $y_L \neq y^*$. When $\lambda = 0$, Proposition 1 implies that if candidate L locates at y_L, there exists a $y_R \in Y$ such that $s_L(y_L, y_R; 0) < \frac{1}{2}$. This implies that $y_L \notin W_L(0)$, proving the result.

(iii) Recall from part (i) that,

$$s_L(y_L, y_R; \lambda) = \int_{v \in \mathbb{R}^J} G_{z|v}\left(h(y_R - v) - h(y_L - v) - \lambda | v\right) g_v(v) \mathrm{d}v$$

It follows that $s_L(y_L, y_R; \lambda)$ can be made arbitrarily large by increasing λ. Since Y^2 is compact and $s_L(y_L, y_R; \lambda)$ is continuous in (y_L, y_R), it follows that there exists a $\bar{\lambda}$ such that for all $\lambda \geq \bar{\lambda}$ and $(y_L, y_R) \in Y^2$, $s_L(y_L, y_R; \lambda) \geq \frac{1}{2}$. Hence, we have $W_L(\lambda) = Y$ for all $\lambda \geq \bar{\lambda}$.

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