

Assignment 1

1. Suppose that $A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ 1 & 5 \\ 6 & 2 \end{bmatrix}$. Compute AB , $A'B'$, and BA .
2. Verify the property $\lambda(v + w) = \lambda v + \lambda w$ where λ is a scalar and v and w are vectors of equal dimension using only the definitions of vector addition ($[v + w]_i = v_i + w_i$) and multiplying a scalar by a vector ($[\lambda v]_i = \lambda v_i$).
3. Verify that $AI = A$ where the number of columns of A is equal to the dimension of the identity matrix I using only the definition of the identity matrix ($I_{ij} = 1$ for $i = j$ and $I_{ij} = 0$ for $i \neq j$) and the definition of matrix multiplication ($[AB]_{ij} = \sum_{k=1}^r A_{ik} B_{kj}$).
4. Suppose that the dimensions of A and B are such that both AB and BA can be formed. Prove that $tr(AB) = tr(BA)$.
5. Consider a two by two matrix, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Verify that the inverse of A is given by the formula, $A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$.
6. What is the rank of the matrix,
 $\begin{pmatrix} 1 & 2 & -1 \\ 2 & 2 & 3 \\ 4 & 6 & 1 \end{pmatrix}$

Is this matrix singular?

7. Calculate the eigenvalues and the eigenvectors of $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Verify that for this

matrix we have $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = V\Lambda V^{-1}$ where V is the matrix of eigenvectors, with each

eigenvector in a column of V , and Λ is a diagonal matrix with the diagonal elements equal to the eigenvalues.

8. Compute the eigenvalues of $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}$. What is the rank of A ? (justify your

answer with reference to the eigenvalues). Verify that the trace of A is equal to the sum of the eigenvalues and that the determinant is equal to the product of the eigenvalues.

(Hint: The formula for the determinant of a 3x3 matrix is given by,

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

9. Prove that $(A')^{-1} = (A^{-1})'$. (Hint: start with the equation $(A')^{-1}A' = I$ and use the fact that $I' = I$).

10. In class, we showed that the multivariate normal distribution is given by,

$$f(x; \mu, \Omega) = \frac{1}{(2\pi)^{K/2} \det(\Omega)^{1/2}} e^{-\frac{1}{2}(x-\mu)'\Omega^{-1}(x-\mu)}$$

Verify that this formula is equivalent to the formula for the bivariate normal distribution,

$$f(x_1, x_2; \mu, \Omega) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right)}$$

where,

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \Omega = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

11. Consider the two-by-two variance covariance matrix, $\Omega = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$.

What is the Cholesky decomposition of $\Omega = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$? (Hint: denote the

Cholesky decomposition by $L = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix}$ and solve, $LL' = \Omega$ for L_{11} , L_{21} , and L_{22}).

12. Suppose that A is symmetric positive definite with Cholesky decomposition $A = LL'$. Prove that the Cholesky decomposition of A^{-1} is given by $A^{-1} = (L^{-1})'(L^{-1})$.

13. Recall that $\text{Var}(AX + b) = A\text{Var}(X)A'$. Verify that

$\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2) + 2\text{Cov}(X_1, X_2)$ by writing $X_1 + X_2$ in the form $AX + b$ and applying the formula.

14. Consider the OLS estimator, $\hat{\beta} = \left(\frac{1}{N} \sum_{n=1}^N x_n x_n' \right)^{-1} \left(\frac{1}{N} \sum_{n=1}^N x_n y_n \right)$. Verify that when

$K = 2$, we have, $\hat{\beta} = \frac{1}{x^2 - \bar{x}^2} \begin{bmatrix} \overline{x^2 y} - \bar{x} \bar{xy} \\ \overline{xy} - \bar{x} \bar{y} \end{bmatrix}$. Then verify that this corresponds with the

expressions for \hat{a} and \hat{b} that we derived in class.