Assignment 1

1. Suppose that $A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ 1 & 5 \\ 6 & 2 \end{bmatrix}$. Compute AB, $A'B'$, and BA.

2. Verify the property $\lambda(v+w) = \lambda v + \lambda w$ where λ is a scalar and v and w are vectors of equal dimension using only the definitions of vector addition ($[v+w]_i = v_i + w_i$) and multiplying a scalar by a vector ($[\lambda v]_i = \lambda v_i$).

3. Verify that $AI = A$ where the number of columns of A is equal to the dimension of the identity matrix I using only the definition of the identify matrix ($I_{ij} = 1$ for $i = j$ and $I_{ij} = 0$ for $i \neq j$) and the definition of matrix multiplication ($[AB]_{ij} = \sum_{k=1}^{r} A_{ik} B_{kj}$).

4. Suppose that the dimensions of A and B are such that both AB and BA can be formed. Prove that $tr(AB) = tr(BA)$

5. Consider a two by two matrix, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Verify that the inverse of A is given by the formula, $A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$

6. What is the rank of the matrix,

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 2 & 3 \\ 4 & 6 & 1 \end{pmatrix}$$

Is this matrix singular?

7. Calculate the eigenvalues and the eigenvectors of $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Verify that for this matrix we have $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = V\Lambda V^{-1}$ where V is the matrix of eigenvectors, with each eigenvector in a column of V, and Λ is a diagonal matrix with the diagonal elements equal to the eigenvalues.

8. Prove that $(A')^{-1} = (A^{-1})'$. (Hint: start with the equation $(A')^{-1}A' = I$ and use the fact that $I' = I$).

9. In class, we showed that the multivariate normal distribution is given by,

$$f(x;\mu,\Omega) = \frac{1}{(2\pi)^{K/2}\det(\Omega)^{1/2}} e^{-\frac{1}{2}(x-\mu)'\Omega^{-1}(x-\mu)}$$

Verify that this formula is equivalent to the formula for the bivariate normal distribution,

$$f(x_1,x_2;\mu,\Omega) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left((\frac{x_1-\mu_1}{\sigma_1})^2+(\frac{x_2-\mu_2}{\sigma_2})^2-2\rho(\frac{x_1-\mu_1}{\sigma_1})(\frac{x_2-\mu_2}{\sigma_2})\right)}$$

where,

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \qquad \Omega = \begin{bmatrix} \sigma_1{}^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2{}^2 \end{bmatrix}$$

10. Consider the two-by-two variance covariance matrix, $\Omega = \begin{bmatrix} \sigma_1{}^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2{}^2 \end{bmatrix}$.

What is the Cholesky decomposition of $\Omega = \begin{bmatrix} \sigma_1{}^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2{}^2 \end{bmatrix}$? (Hint: denote the Cholesky decomposition by $L = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix}$ and solve, $LL' = \Omega$ for L_{11}, L_{21}, and L_{22}).

11. Suppose that A is symmetric positive definite with Cholesky decomposition $A = LL'$. Prove that the Cholesky decomposition of A^{-1} is given by $A^{-1} = (L^{-1})'(L^{-1})$.

12. Recall that $Var(AX + b) = AVar(X)A'$. Verify that $Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)$ by writing $X_1 + X_2$ in the form $AX + b$ and applying the formula.

13. Consider the OLS estimator, $\hat{\beta} = \left(\frac{1}{N}\sum_{n=1}^{N} x_n x_n'\right)^{-1}\left(\frac{1}{N}\sum_{n=1}^{N} x_n y_n\right)$. Verify that when $K = 2$, we have, $\hat{\beta} = \frac{1}{\overline{x^2} - \overline{x}^2}\begin{bmatrix} \overline{x^2}\,\overline{y} - \overline{x} * \overline{xy} \\ \overline{xy} - \overline{x} * \overline{y} \end{bmatrix}$. Then verify that this corresponds with the expressions for $\hat{b} = \frac{\overline{xy} - \overline{x} * \overline{y}}{\overline{x^2} - \overline{x}^2}$ and $\hat{a} = \overline{y} - \hat{b}\overline{x}$ that we derived in class.